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ON THE CONNECTIVITY OF PSEUDORANDOM GRAPHS THROUGH AN EIGENVALUE

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ABSTRACT

In this paper we discuss some theorems about pseudorandom graphs by Krivelevitch and Sudakov [1] and generalized them on the basis of edge-connectivity, vertex-connectivity of graph G through an eigenvalue. Here, we give a solution for the question raised in [1], for what are the possible values of second Laplacian eigenvalue of graphs on n vertices with an cut-vertex?

KEYWORDS: Eigenvalues, Edge-Connectivity, Vertex-Connectivity and Pseudorandom Graphs

1.0 INTRODUCTION

Pseudo-random graphs are graphs which behave like random graphs. Random graphs have proven to be one of the most important and fruitful concepts in modern Combinatorics and Theoretical Computer Science. Pseudo random graphs are modeled after truly random graphs, and therefore mastering the edge distribution in random graphs can provide the most useful insight on what can be expected from pseudo-random graphs

Speaking very informally, a pseudo-random graph $G = (V, E)$ is a graph that behaves like a truly random graph $G(|V|, p)$ of the same edge density $p = |E|/\binom{|V|}{2}$. But it is not very informative, there are quite a few possible graph parameters that can potentially serve for comparing pseudo-random and random graphs. Probably the most important characteristics of a truly random graph is its edge distribution. We can thus make a significant step forward and say that a pseudo-random graph is a graph with edge distribution resembling the one of a truly random graph with the same edge density. Still, the quantitative measure of this resemblance remains to be introduced.

Andrew Thomason launched systematic research on examples and applications of pseudo-random graphs in his two papers [2, 3]. Thomason introduced the notion of jumbled graphs, enabling to measure in quantitative terms the similarity between the edge distributions of pseudorandom and truly random graphs. He also supplied several examples of pseudo-random graphs and discussed many of their properties. Thomason's papers undoubtedly defined directions of future research for many years. Another cornerstone contribution belongs to Chung, Graham and Wilson [4] who in 1989 showed that many properties of different nature are in certain sense equivalent to the notion of pseudo-randomness, defined using the edge distribution. This fundamental result opened many new horizons by showing additional facets of pseudo-randomness.

Our paper is designed as follows; we have studied a survey paper [1], presented some of the theorems in the form of propositions and generalized, proved through an eigenvalue. For basic facts and result about pseudorandom graphs we refer [1], and our earlier paper [5]. In the first section we have given brief introduction, definitions and some results about the pseudorandom graphs.

Section 1.1 and 1.2 deals with edge-connectivity, vertex-connectivity of pseudorandom graphs respectively. In the section we give conclusion.

Here we need the following definitions and theorems which will help in construction of the proofs.

Theorem 1.1 (Whitney Theorem): If G is a simple graph, then $\kappa(G) \leq \kappa'(G) \leq \delta(G)$

Theorem 1.2 (Tutte's 1-Factor Theorem): A graph G has a perfect matching if and only if the number of odd components of $G - S \leq |S|$ for every subset S of vertices.

Definition: Random Graph 1.3: A random graph $G(n, p)$ is a probability space of all labeled graphs on n vertices {1, 2, . . .n} where {i, j}is an edge of $G(n, p)$ with probability $p = p(n)$, independently of any other edges, for 1 ≤i < j ≤ n. Equivalently, the probability of a graph G = (V, E) with n vertices and e edges in G(n p) is $p^e(1-p)^{\binom{n}{2}-e}$. We observe that for p = 1/2 the probability of every graph is the same and for p > 1/2 the probability of a graph G_1 with more edges than another graph G_2 is higher. (And the probability of G_1 is smaller than the probability of G_2 if p < 1/2.) Almost all properties hold for all $G \in G(n, p)$ has property p tends to one as n tends to infinity. The following definition and theorem are concerned about the edge distribution of random graphs.

Definition 1.4: (e(U W)) The number of edges between two disjoint subsets U, W of vertices is denoted by e(U W). More generally, we define $e(U\ W) = \sum_{u \in U} \sum_{v \in V : u \sim v} 1$.

Note that if we have an edge e with both endpoints in U∩W then this edge is counted twice in $e(U\ W)$.

Theorem 1.5: (Random Graph: Edge Distribution ([1], p.4)): Let $p = p(n) \leq 0.9$. Then for every two (not necessarily disjoint) subsets U, W of vertices $|e(U\ W) - p|U||W|| = O\left(\sqrt{|U|\,|W|\,np}\right)$.

For almost all $G \in G(n, p)$. The expected degree of a vertex v in a random graph $G(n, p)$ is the same for all v. We consider in the following an extended random graph model for a general degree distribution.

Definition 1.6: (Random Graph with Given Degree Distribution): Given $w = w_1, w_2, \ldots w_n$ a sequence. A random graph with given degree distribution $G(w)$ is a probability space of all labeled graphs on n vertices {1, 2,.,., n} where {i, j} is an edge of $G(w)$ with probability $w_i w_j \rho$, independently of any other edges, for 1 ≤ i < j ≤ n. Where ρ plays the role of a normalization factor, i.e. $\rho = (\sum_{i=1}^{n} w_i)^{-1}$.

The expected degree of the vertex i is w_i. The above definition of a random graph with given degree distribution comes from Chung and Lu [6]. There are some other definitions for random graphs with given degrees (interested reader can see Watts, D. J., and et. all. [7]). Next we define the concept of pseudo-random graphs by means of the eigenvalues of a graph. Then we can make some statements about the edge-distribution of G. In the following we have

Definition 1.7: (Second Adjacency Eigenvalue): The second adjacency eigenvalue λ(A(G)) is defined as

$$\lambda(A(G)) = \max\{-\lambda_1(A(G)),\ \lambda_{n-1}(A(G))\}$$

$$= \max_{i \neq n} |\lambda_i(A(G))|$$

This will be our link to spectral graph theory.

Definition 1.8: ((n, d, λ)-Graph): A (n, d, λ)-graph is a d-regular graph on n vertices with second adjacency eigenvalue at most λ .

Proposition 1.9: ([6], Chapter 9): Let G = (V, E) be a (n, d, λ)-graph. Then for every subset B of V we have

$$\sum_{v \in V}\left(|N_B(v)| - \frac{|B|d}{n}\right)^2 \leq \lambda \frac{|B|(n-|B|)}{n}$$

where $N_B(v)$ denotes the set of all neighbors in $B$of v.

Theorem 1.10: ((n, d, λ)-Graph: Edge Distribution ([6], Chapter 9)): Let G be a (n, d, λ)-graph Then for every two subsets B, C of vertices, we have $\left|e(B\ C) - \frac{d|B||C|}{n}\right| \leq \lambda\sqrt{|B||C|}$

If for a graph G the second adjacency eigenvalue $\lambda\left(A(G)\right)$ is small then the edge distribution of G is almost the same as for the random graph $G(n, \frac{d}{n})$ by the above theorem graph G is very pseudo-random.

There is a generalization of Theorem 1.2.8 with an improved error term.

Theorem 1.11: ((n, d, λ)-Graph: Edge Distribution [2]): Let G be a (n, d, λ)-graph Then for every two subsets $, W \subseteq V$,

$$\left|e(U, W) - \frac{d|U||W|}{n}\right| \leq \lambda \sqrt{|U||W|\left(1-\frac{|U|}{n}\right)\left(1-\frac{|W|}{n}\right)}$$

Definition 1.12: (**Second Laplacian Eigenvalue):** The second Laplacian eigenvalue is defined

as $\lambda\left(\mathcal{L}(G)\right) = \max_{i \neq n} \left|1 - \lambda_i\left(\mathcal{L}(G)\right)\right|$

$= \max\{\lambda_n\left(\mathcal{L}(G)\right) - 1,\ 1 - \lambda_2\left(\mathcal{L}(G)\right)\}$

Lemma 1.13: Let G be a d-regular graph. Then the second adjacency eigenvalue is d times the second Laplacian eigenvalue, i.e. $\lambda\left(A(G)\right) = d.\lambda(\mathcal{L}(G))$.

Definition 1.14: (**Volume):** The volume of a subset U of the vertices is defined as

$$\mathrm{vol}(U) = \sum_{j \in U} d_j$$

1.1 Edge Connectivity

Proposition 1.1.1 ([1], p.25): Let G be an (n, d, λ)-graph with $G - \lambda \geq 2$ Then G is d-edge-connected. When n is even, it has a perfect matching.

Proof: We will only prove the second statement by using Tutte's condition (see Theorem 1.2). Since n is even, we need to prove that for every nonempty set S of vertices the induced graph $G[V-S]$ has at most $|S|$ connected components of odd size. From the first point of the statement, we know that G is d-edge-connected, so

$e(S,\bar{S}) \geq \gamma d$

Where γ is the number of components in $G[V-S]$. On the other hand there are at most $d|S|$ edges incident with vertices in S: $e(S,\bar{S}) \geq d|S|$. Therefore $G[V-S]$ has at most $|S|$ connected components and hence G contains a perfect matching.

Definition: 1.1.2: ($\bar{d}_U$): Let G = (V, E) be a graph. Then we write

$$\bar{d}_U = \frac{\mathrm{vol(U)}}{|\mathrm{U}|} = \frac{\sum_{u\in U} d_i}{|U|}$$

for the average degree of some set λ. We write $\bar{d}$ if we mean $\bar{d}_V$.

Theorem 1.1.3: Let U be a subset of vertices. Then$\delta \leq \bar{d} \leq \Delta$

Proof: We have

$$\bar{d}_U = \frac{\mathrm{vol(U)}}{|\mathrm{U}|} = \frac{\sum_{u\in U} d_i}{|U|} \leq \frac{\sum_{u\in U} \Delta}{|U|} \leq \Delta$$

In fact, Δ is the maximum of all of the average degrees$\bar{d}_U$. Similarly bound each degree by the minimum degree and get the lower bound.

i.e.

$$\bar{d}_U = \frac{vol(U)}{|U|} = \frac{\sum_{u\in U} d_i}{|U|} \geq \frac{\sum_{u\in U} \delta}{|U|} \geq \delta$$

Clearly, we have $\frac{vol(U)}{|U|} = |U|\bar{d}_U$ Thus we can rewrite theorem 1.1.3 in the following form.

Theorem 1.1.4: (**Second Laplacian Eigenvalue: Edge-Distribution ([6], p.72)):** Let G be a graph on n vertices with normalized Laplacian $\mathcal{L}(\mathrm{G})$ and second Laplacian eigenvalue λ. Then for any two subsets X and Y of vertices

$$\left|\mathrm{e(X,Y)} - \frac{\mathrm{vol(X)vol(Y)}}{\mathrm{vol(G)}}\right| \leq \lambda\sqrt{\mathrm{vol(X)vol(Y)}}$$

There is a slightly stronger result:

Theorem 1.1.5: Let G be a graph on n vertices with normalized Laplacian $\mathcal{L}(\mathrm{G})$ and second Laplacian eigenvalue λ. Then for any two subsets X and Y of vertices

$$\left|\mathrm{e(X,Y)} - \frac{\bar{d}_X|\mathrm{X}|\bar{d}_Y|\mathrm{Y}|}{\bar{d}_n}\right| \leq \lambda\sqrt{\bar{d}_X|\mathrm{X}|\bar{d}_Y|\mathrm{Y}|}$$

where G = (V, E) is a graph on n vertices with second Laplacian eigenvalue λ and $X, Y \subseteq V$.

Theorem 1.1.6 (δ-Edge-Connected): Let G be a graph on n vertices with second Laplacian eigenvalue λ and minimum degree δ such that $\delta(1-\lambda) \geq 2$ Then G is δ-edge-connected, even $\kappa'(G) = \delta$.

Proof: Let $U \subseteq V(G)$ with $|U| \leq n/2$.

We want to show that there are at least $\delta(1-\lambda)$ edges between U and $\bar{U}$.

Case 1: $1 \leq |U| \leq \bar{d}_U = \frac{vol(U)}{|U|}$

$e(U,\bar{U}) = \sum_{v\in U} d_v - e(U,\ U)$

$\geq |U|\bar{d}_U - |U|(|U|-1)$

$= |U|(\bar{d}_U - |U| + 1) \geq \bar{d}_U \geq \delta$

Case 2: $\bar{d}_U \leq |U| \leq n/2$

By using theorem 1.3.7 [2] $e(U,\bar{U}) = \frac{\bar{d}_U |U| \bar{d}_{\bar{U}} (n-|U|)}{\bar{d}_n} - \lambda \frac{\bar{d}_U |U| \bar{d}_{\bar{U}} (n-|U|)}{\bar{d}_n}$

$$= \frac{\bar{d}_U \bar{d}_{\bar{U}}}{\bar{d}} . \frac{|U|\,(n-|U|)}{n} . (1-\lambda)$$

$$\geq \frac{\bar{d}_U \bar{d}_{\bar{U}}}{\bar{d}} (1-\lambda) . \frac{1}{2} . |U|$$

Now we will look closer at the fraction $\frac{\bar{d}_U \bar{d}_{\bar{U}}}{\bar{d}}$.

Note that,

$$\bar{d} = \frac{vol(G)}{|n|} = \frac{vol(U) + vol(\bar{U})}{|n|}$$

$$= \frac{|U|\ \frac{vol(U)}{|U|}(n-|U|\ ) + \frac{vol(\bar{U})}{n-|U|}}{|n|}$$

$$\bar{d} \leq \max\left\{\frac{vol(U)}{|U|}, \frac{vol(\bar{U})}{n-|U|}\right\}$$

$$= \max\{\bar{d}_U, \bar{d}_{\bar{U}}\}.$$

Thus

$$e(U,\bar{U}) \geq \frac{\bar{d}_U \bar{d}_{\bar{U}}}{\max\{\bar{d}_U, \bar{d}_{\bar{U}}\}} (1-\lambda) \frac{1}{2} |U|$$

$$s \geq \delta(1-\lambda) \frac{|U|}{2}$$

$$\geq |U| \geq \bar{d}_U \geq \delta.$$

So this shows $\kappa'(G) \geq \delta$. To get equality, we take a vertex v such that $d_v = \delta$ and delete all edges from v (see also Whitney's Theorem 1.1).

Here we have actually proved the following stronger statement:

Theorem 1.1.7: Let G = (V, E) be a graph on n vertices with second Laplacian eigenvalue and $U \subseteq V$. If $|U| \leq \frac{n}{2}$, $\bar{d}_U(1-\lambda) \geq 2$ then $e(U, \bar{U}) \geq \bar{d}_U$.

Example 1.1.8: from the above theorem we may conclude that, if a graph is not δ- edge-connected then this graph is not very pseudo-random. Let n > 2 be an integer. We take two copies of the complete graph K_n and connect them by one edge, denote the obtained graph by G. Then $\kappa'(G) = 1$ but the minimum degree is $n-1$. So the edge-connectivity is much smaller than the minimum degree. This gives the information that this graph is not very pseudo-random. More precisely, above Theorem 1.1.3 give us $\lambda(G) > 1 - \frac{2}{n-1}$ which will even tend to 1 as $n \to \infty$.

1.2 Vertex Connectivity

Here we will use the notation of the average degree of some set U and the theorem 1.1.3.

Proposition 1.2.1 ([1], p.23): Let G be an (n, d, λ)--graph with Then the vertex-connectivity of G satisfies

$$\kappa(G) \geq d - 36\frac{\lambda^2}{d}$$

Theorem 1.2.2 (Vertex-Connectivity): Let G be a graph on n vertices with second Laplacian eigenvalue λ such that $\Delta \leq n/2$. Then $\kappa(G) > \delta - 36\lambda^2\Delta$

Proof: Without loss of generality, we can assume

$$\kappa(G) > \frac{1}{6}\sqrt{\frac{\delta}{\Delta}} \quad (1)$$

Because otherwise the right hand side would be negative and thus the statement is trivially true.

We will now proceed to prove by using indirect proof. Assume that their exist $S \subseteq V(G)$ such that $|S| \leq \delta - 36\lambda^2\Delta$ and $G[V-S]$ disconnected. Denote by U the smallest component of $G[V-S]$ and $W = V - S$ the rest.

Then

$$W = n - |S| - \left| |U| \geq \frac{n-|S|}{2} \geq \frac{n-\delta}{2} \geq \frac{n-\Delta}{2} \geq \frac{n}{4} \right| \quad (2)$$

Since all neighbors of a vertex u from U are contained in $S \cup U$, we get

$$d_u < |N(u) \cup \{u\}| \leq |S \cup U| = |S| + |U|.$$

So

$$|S| + |U| > \bar{d}_U \geq \delta$$

Implies that, $|U| > 36\lambda^2\Delta$ (3)

Theorem 1.1.4 gives us

$$\left| e(U,W) - \frac{\bar{d}_U |U| \bar{d}_W |W|}{\bar{d}_n} \right| \leq \lambda \sqrt{\bar{d}_U |U| \bar{d}_W |W|}$$

and $e(U,W) = 0$. This implies that

$$|U| \leq \frac{\lambda^2 \bar{d}^2 n^2}{\bar{d}_U \bar{d}_W |W|} = \lambda \frac{\bar{d}}{\bar{d}_W} \cdot \frac{n}{|W|} \cdot \frac{\lambda \bar{d} n}{\bar{d}_U}$$

From (1) and (2), we have

$$\leq \frac{1}{6} \sqrt{\frac{\delta}{\Delta}} \cdot \frac{\Delta}{\delta} \cdot 4 \cdot \frac{\lambda \bar{d} n}{\bar{d}_U} < \sqrt{\frac{\Delta \lambda \bar{d} n}{\delta \bar{d}_U}} \tag{4}$$

Now we double count the number of edges between U and S. On the one hand we have:

$$e(U,S) = \bar{d}_U |U| - e(U,U)$$

from theorem 1.1.4. We have

$$\geq \bar{d}_U |U| - \frac{\bar{d}_U^2}{\bar{d} n} - \lambda \bar{d}_U |U|$$

From equation (4)

$$> |U| \left(\bar{d}_U - \frac{2\lambda \bar{d}_U \sqrt{\Delta}}{\sqrt{\delta}} \right)$$

From equation (1) we have $-\frac{2}{3} |U| \bar{d}_U$.

On the other hand, $e(U,S) \leq \frac{\bar{d}_U |U| \bar{d}_S |S|}{\bar{d}_n} + \lambda \sqrt{\bar{d}_U |U| \bar{d}_S |S|}$

$$\leq \frac{\bar{d}_S |S|}{n} \bar{d}_U |U| + \lambda \sqrt{\bar{d}_U |U| \bar{d}_S \delta}$$

Since $\Delta \leq n/2$, implies

$$\leq \frac{\bar{d}_U |U|}{2} + \frac{\lambda \bar{d}_U |U| \sqrt{\bar{d}_S \delta}}{\sqrt{\bar{d}_U |U|}}$$

From equation (3),

$$< \frac{\bar{d}_U |U|}{2} + \frac{\lambda \bar{d}_U |U|}{6\lambda} \cdot \frac{\sqrt{\bar{d}_S \delta}}{\bar{d}_U \Delta}$$

$$\leq \frac{\bar{d}_U |U|}{2} + \frac{\lambda \bar{d}_U |U|}{6\lambda}.$$

Because $\frac{\sqrt{\bar{d}_S \delta}}{\bar{d}_U \Delta} \leq 1 \leq \frac{2\bar{d}_U |U|}{3}$

This is a contradiction and so therefore theorem is proved.

Next we answer for the question, what are the possible values for the second Laplacian eigenvalue of graphs on n vertices with an cut-vertex?

The star S_n has an cut-vertex and is bipartite. Thus by example 1.3.2 of [2] the second Laplacian eigenvalue of S_n is 1. This is the maximum value for the second Laplacian eigenvalue.

1.3 CONCLUSIONS

In this paper we have generalized some theorem on the basis of edge-connectivity, vertex-connectivity of pseudorandom graphs and proved through an eigenvalue. We borrowed necessary theorem from [1] and presented in the form of proposition. We prove the more stronger theorem (1.1.7), gives us information, that if G is δ-edge connected than it is very pseudorandom otherwise not. Finally we answered for the question raised in [1], the possible values for the second Laplacian eigenvalue of graphs on n vertices with an cut-vertex. The star S_n has an cut-vertex and is bipartite. Thus by example 1.3.2 of [5] the second Laplacian eigenvalue of S_n is 1. This is the maximum value for the second Laplacian eigenvalue. In future we are looking at subgraph of pseudorandom graphs through an eigenvalues.

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